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Normal frames and the validity of the equivalence principle: I. Cases in a neighbourhood and at a point

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Abstract. A treatment in a neighbourhood and at a point of the equivalence principle on the basis of derivations of the tensor algebra over a manifold is given. Necessary and sufficient conditions are given for the existence of local bases, called normal frames, in which the components of derivations vanish in a neighbourhood or at a point. These frames (bases), if any, are explicitly described and the problem of their holonomicity is considered. In particular, the results obtained concern symmetric as well as non-symmetric linear connections.

1. Introduction

Usually in a local frame (basis) the gravitational field strength is identified with the components of a linear connection which may be with or without torsion (e.g., the Riemannian one in general relativity [1] or the one in Riemann-Cartan spacetimes [2]). This linear connection must be compatible with the equivalence principle in a sense that there must exist ‘local’ inertial, called also Lorentz, frames of reference (bases) in which the gravity field strength is ‘locally’ transformed to zero. Mathematically this means the existence of special ‘local’ basis (or bases) in which the components of the connection vanish ‘locally’. Above the words ‘local’ and ‘locally’ are in quotes as they are not well defined here, a usual fact for the physical literature [1], where they often mean ‘infinitesimal surrounding of a fixed point of spacetime’ [2]. The strict meaning of ‘locally’ may be at a point, in a neighbourhood, along a path (curve) or on some other submanifold of the spacetime. The present paper deals with the first two of these meanings of ‘locally’, in which cases the equivalence principle is considered.

The existence of (local) bases or coordinates in which the components of linear connections [3, 4] vanish at a point [2, 4–8], along a curve [5, 8] or in a neighbourhood [5, 7, 8] have been considered. But with very rare exceptions (see, e.g., [2]) in the literature only the torsion-free case has been investigated. The present work, which is a revised version of [9], generalizes these problems to the case of arbitrary derivations of the tensor algebra over a differentiable manifold (see either [4] or section 2 of the present paper) whose curvature and torsion are not *a priori* restricted.

Physically the goal of the paper is to show that gravity theories, based, first of all, on linear connections, are compatible with the equivalence principle because of their underlying mathematical structure.

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Mathematically the main purpose of this work is to find necessary and sufficient conditions for the existence of local bases (coordinates) in which the components of a derivation (of the tensor algebra over a manifold) vanish. If such special bases (frames, called normal) exist, the problem of their holonomicity [5] is considered.

In section 2 notation and some definitions are introduced. Section 3 deals with the above problems in a neighbourhood and section 4 investigates them at a single point. Their connection with the equivalence principle is shown in section 5. Section 6 contains some concluding remarks.

2. Derivations, their components, curvature and torsion

Let D be a derivation of the tensor algebra over a manifold M [3, 4]. By [4, proposition 3.3 of chapter I] there exists a unique vector field X and a unique tensor field S of type $(1, 1)$ such that $D = L_X + S$. Here L_X is the Lie derivative along X [3, 4] and S is considered as a derivation of the tensor algebra over M [4].

If S is a map from the set of C^1 vector fields into the tensor fields of type (1,1) and $S : X \mapsto S_X$, then the equation $D^S_X = L_X + S_X$ defines a derivation of the tensor algebra over M for any C^1 vector field X [4]. Such a derivation will be called an S-derivation along X and denoted for brevity simply by D_X. An S-derivation is a map D such that $D : X \mapsto D_X$, where D_X is an S-derivation along X.

Evidently (see the above-cited proposition from [4]), every derivation of the tensor algebra is an S-derivation along some fixed vector field and vice versa.

In this work we shall not be interested on the concrete dependence of S_X in $D_X = L_X + S_X$ on X. In it as an example of S-derivation only the covariant differentiation will be considered. It turns out to be an S-derivation linear over functions with respect to the vectors along which it acts.

Let $\{E_i, i = 1, \ldots, n := \dim(M)\}$ be a (coordinate or not [5, 6]) local basis (frame) of vector fields in the tangent to M bundle. It is holonomic (anholonomic) if the vectors $E_1, \ldots, E_n$ commute (do not commute) [5, 6]. Using the explicit action of L_X and S_X on tensor fields [4] one can easily deduce the explicit form of the local components of $D_X T$ for any C^1 tensor field T. In particular, we have

$$D_X(E_j) = (W_X)^i_j E_i. \tag{1}$$

Here and below all Latin indices, perhaps with some super- or subscripts, run from 1 to $n := \dim(M)$, the usual summation rule on indices repeated on different levels is assumed, and $(W_X)^i_j := (S_X)^i_j - E_j(X^i) + C^i_{kj} X^k$ where $X(f)$ denotes the action of $X = X^k E_k$ on the C^1 scalar function f, i.e. $X(f) := X^k E_k(f)$ and C^i_{kj} define the commutators of the basic vector fields by $[E_j, E_k] = C^i_{jk} E_i$. We call $(W_X)^i_j$ the *components* of D_X.

The change $\{E_i\} \mapsto \{E'_m := A^i_m E_i\}$, $A := [A^i_m]$ being a non-degenerate matrix function, implies the transformation of $(W_X)^i_j$ into (see equation (1)) $(W'_X)^m_l = (A^{-1})^m_i A^j_l (W_X)^i_j + (A^{-1})^m_i X(A^i_l)$. Introducing the matrices $W_X := [(W_X)^i_j]$ and $W'_X := [(W'_X)^m_l]$ and putting $X(A) := X^k E_k(A) = [X^k E_k(A^i_m)]$, we get

$$W'_X = A^{-1}\{W_X A + X(A)\}. \tag{2}$$

If ∇ is a linear connection with local components Γ^i_{jk} (see, e.g., [3–5]), then $\nabla_X(E_j) = (\Gamma^i_{jk} X^k) E_i$ [4]. Hence, we see from (1) that D_X is a covariant differentiation along X iff

$$(W_X)^i_j = \Gamma^i_{jk} X^k \tag{3}$$

for some functions Γ^i_{jk}. Due to $D^S_X = L_X + S_X$ a linear connection ∇ is characterized by the map $S : X \mapsto S_X = \Sigma_X$, $\Sigma_X(Y) := \nabla_X(Y) - [X, Y]$, $[X, Y] = L_X Y$ being the commutator of the vector fields X and Y [4].

Let D be an S-derivation and X, Y and Z be vector fields. The *curvature operator* R^D, if D is a C^1 derivation (i.e. $(W_X)^i_j$ are C^1 functions), and the *torsion operator* T^D of D are, respectively, $R^D(X, Y) := D_X D_Y - D_Y D_X - D_{[X,Y]}$ and $T^D(X, Y) := D_X Y - D_Y X - [X, Y]$. The S-derivation D is *flat* ($\equiv$ *curvature free*) or *torsion free* if, respectively, $R^D = 0$ or $T^D = 0$ (cf [4]).

For a linear connection ∇ due to (3), we have $(R^\nabla(X, Y))^i_j = R^i_{jkl} X^k Y^l$, $(T^\nabla(X, Y))^i = T^i_{kl} X^k Y^l$, where [4] R^i_{jkl} and T^i_{kl} are the components, respectively, of the curvature and torsion tensors of ∇.

Moreover, we shall look for special bases $\{E'_m\}$ in which the components W'_X of an S-derivation D vanish along some or all vector fields X. For this purpose we have to solve (2) with respect to A under certain conditions. If such bases (frames) exist, they will be called *normal* (sometimes geodesic or Riemannian [5, 10]), as they are so named in the theory of linear connections [4].

3. Normal frames for derivations in a neighbourhood

In this section we shall solve the problems of existence, uniqueness and holonomicity of basis or bases $\{E'_m\}$ in which the components of a given (S-)derivation vanish in some neighbourhood U. Such frames will be called *normal* in U.

Proposition 3.1. In U for an S-derivation D there exists a basis $\{E'_m\}$ such that $W'_X = 0$ for every X if and only if in U the S-derivation D is a flat linear connection, i.e. iff D_X is a covariant differentiation along X with $R^i_{\mathrm{jkl}} = 0$.

Proof. Let us fix a basis $\{E_i\}$ in U. The existence of $\{E'_m\}$ with $W'_X = 0$, due to (2), implies $W_X = -(X(A))A^{-1}$, i.e. $(W_X)^i_j = -[X^k(E_k(A^i_m))](A^{-1})^m_j$ which by (3), means that D is a linear connection with local components $\Gamma^i_{jk} = -(E_k(A^i_m))(A^{-1})^m_j$. Putting $W_X = -(X(A))A^{-1}$ and using $X(A^{-1}) = -A^{-1}(X(A))A^{-1}$, we get $R^D = R^\nabla = 0$.

Conversely, let D be a flat linear connection in U. Let $\{E^0_i\}$ be a basis at $x_0 \in U$. Define the vector field E'_i so that its value $E'_i|_x$ at $x \in U$ is obtained from E^0_i by the parallel translation (transport), generated by D [3, 4], from x_0 to x. As D is a flat linear connection, $E'_i|_x$ does not depend on the path of transport and the vector fields $\{E'_i\}$ are linearly independent [4–6], i.e. they form a basis on U. It is holonomic iff D is torsion free on U [5, 6]. By definition of a parallel translation, the vectors of the basis $\{E'_i\}$ satisfy $D_X E'_i = 0$, which, when combined with (1), implies $W'_X = 0$. □

The main consequence of proposition 3.1 is that the (flat) linear connections are the only S-derivations for which normal frames exist in neighbourhoods. These frames, if any, are holonomic iff the derivation is torsion free [5, 6]. From (2) one finds that they are connected by linear transformations with constant coefficients. By proposition 3.1 a necessary condition for the existence of the considered special bases for an S-derivation D is its flatness, i.e. $R^D = 0$.

Let us turn now to the above-considered problems for S-derivations D_X along a *fixed* vector field X when $X|_x \neq 0$ for $x \in U$.

If $\{E'_m = A^i_m E_i\}$ is a basis with $W'_X = 0$, then by (2) its existence is equivalent to that of $A := [A^i_m]$ obeying $W_X A + X(A) = 0$. As X is fixed, the values of A at different points are connected through the last equation iff the points lie on one and the same integral curve of X. Let $\gamma_y : J \to M$, J being a $\mathbb{R}$-interval, be the integral curve for X passing through $y \in M$, i.e. $\gamma_y(s_0) = y$ and $\dot{\gamma}_y(s) = X\big|_{\gamma_y(s)}$, $\dot{\gamma}_y$ being the tangent to γ_y vector field, for $s \in J$ and a fixed $s_0 \in$J. Along γ_y the equation $W_X A + X(A) = 0$ reduces to $(\mathrm{d}A/\mathrm{d}s)\big|_{\gamma_y(s)} = -W_X(\gamma_y(s))A(\gamma_y(s))$ with general solution $A(\gamma_y(s)) = Y(s, s_0; -W_X \circ \gamma_y)B(\gamma_y)$. Here the non-degenerate matrix B is independent of s and $Y = Y(s, s_0; Z)$, Z being a matrix function of s, is the unique solution of the initial-value problem [11] $\mathrm{d}Y/\mathrm{d}s = ZY$, $Y\big|_{s=s_0} = \mathbb{1}$ with $\mathbb{1}$ being the identity matrix. Thus we have proved:

Proposition 3.2. Let D_X be an S-derivation along a fixed vector field $X(\neq 0)$. Then along the integral curves of X there exist bases in which the components of D_X vanish.

Hence normal frames exist also at any point at which X is defined. Any two such bases are connected by a linear transformation with a matrix A such that $X(A) = 0$ (see equations (1) and (2)).

4. Normal frames for derivations at a point

Here problems analogous to those of the previous section will be investigated in the case describing the behavior of derivations at a given point.

At first we consider S-derivations with respect to a *fixed vector field*, i.e. we shall deal with a *fixed* derivation.

Lemma 4.1. Let

$$A(y) = B - \Gamma_k B(x^k(y) - x^k(x_0)) + B_{kl}(y)(x^k(y) - x^k(x_0))(x^l(y) - x^l(x_0))$$

where $y, x_0 \in M$, $B =$ constant is non-degenerate matrix, i.e. $\det B \neq 0$, Γ_k are independent of y matrices, and the matrices B_{kl} and their first derivatives are bounded when $y \to x_0$. Then there exists a neighbourhood U of x_0 in which the change of the bases $\{\partial/\partial x^i\} \to \{E'_m = A^i_m \partial/\partial x^i\}$ is well defined, i.e. bijective, in U, that is $\det A(y) \neq 0$ for $y \in U$.

Proof. Putting $U' := \{z : 1 + c_k \varepsilon^k(z) > 0\}$ with $\varepsilon^k(y) := x^k(y) - x^k(x_0)$ and $c_k := (\det B)^{-1} \left(\partial \det A(y)/\partial \varepsilon^k(y)\right)_{\varepsilon(y)=0}$, $\varepsilon(y) := \max_k |\varepsilon^k(y)|$, we fined $\det A(y) = (\det B)\det[\mathbb{1} - \Gamma_k \varepsilon^k(y) + \mathrm{O}((\varepsilon(y))^2)] = (\det B)[1 + c_k \varepsilon^k(y) + \mathrm{O}((\varepsilon(y))^2)]$. Hence, using that $\det B \neq 0$, we can choose a neighbourhood $U \subseteq U'$ of x_0 such that $(\det A(y))/(\det B) > 0$ for $y \in U$. (E.g., as $f = O(g)$ for real functions f and g means the existence of $\lambda \in \mathbb{R}_+$ such that $|f| \leqslant \lambda |g|$, we can put $U := \{z : z \in U', (\sum_k |c_k|)\varepsilon(z) + \lambda((\varepsilon(z))^2) \leqslant 1\}$ for some $\lambda \in \mathbb{R}_+$.) Consequently A defines a bijective mapping between $\{\partial/\partial x^i\}$ and E'_m in any such neighbourhood U of x_0. □

Remark. There are different local coordinates $\{y^i\}$ normal at x_0. For instance, one class of normal at x_0 coordinates is separated through the equation $x^i(z) = y^i(z) + b^i_{jk}(y^j(z) - x^j(x_0))(y^k(z) - x^k(x_0))$ for $[b_{jk}] = -W_X/2$. More general classes of normal at x_0 bases and (holonomic) coordinates for fixed X are described in [9, propositions 8 and 9].

Proposition 4.1. Let $x_0 \in M$, X be a vector field with $X|_{x_0} \neq 0$, and D be an S-derivation. Then there exist normal at x_0 local coordinates $\{y^i\}$, i.e. such that $(D_X \partial_{y^i})|_{x_0} = 0$.

Proof. Let $A(z)$, $z \in M$ be defined as in lemma 4.1 for $B = \mathbb{1}$, $B_{kl} = 0$, and $(\Gamma_k)^i_j = -2b^i_{kj} = -2b^i_{jk} \in \mathbb{R}$. Then by lemma 4.1 there is a neighbourhood U of x_0 in which $\{E'_m = A^i_m \partial/\partial x^i\}$ with the matrix $A(z) = \mathbb{1} + \Gamma_k(x^k(z) - x^k(x_0))$ form a field of bases in U. In it, using (2), we find $W'_X(x_0) = W_X(x_0) + \Gamma_k X^k\big|_{x_0}$. As $X\big|_{x_0} \neq 0$ we can choose

$\{x^i\}$ such that $X = \partial/\partial x^1$. Now, partially fixing $\{\Gamma_k\}$ by defining $\Gamma_1 = [-2b^i_{j1}] = W_X$, we get $W'_X(x_0) = 0$. Hence $\{E'_m\}$ is normal at x_0. Besides, it is holonomic at x_0 as $[E'_k, E'_m]\big|_{x_0} = -2(b^j_{km} - b^j_{mk})\partial/\partial x^j\big|_{x_0} \equiv 0$. So, there exist local coordinates $\{y^i\}$ in a neighbourhood V of x_0 such that $E'_k\big|_{x_0} = \partial/\partial x^k\big|_{x_0}$. Evidently, in $V \cap U$ the coordinates $\{y^i\}$ are normal at x_0. □

From equations (1) and (2) we find that a basis $\{E'_m\}$ in which $W'_X(x_0) = 0$ is obtained from $\{\partial/\partial y^i\}$, $\{y^i\}$ defined in the last proof, by a linear transformation with a matrix A such that $(X(A))|_x = 0$. The holonomicity of these normal frames depends on the concrete choice of A.

If $X|_{x_0} = 0$, then a basis $\{E'_m\}$ in which $W'_X(x_0) = 0$ exists if $W_X(x_0) = 0$ in some basis $\{E_i\}$, so then every basis, including the holonomic ones, will have the needed property.

Let us now turn our attention to S-derivations with respect to *arbitrary vector fields*.

The S-derivation D is *linear* at x_0 if for all X and some (and hence any) basis $\{E_i\}$ we have (cf equation (3)) $W_X(x_0) = \Gamma_k X^k(x_0)$, where the Γ_k are constant matrices. This means (3) is valid at x_0, but may not be true at $x \neq x_0$.

Proposition 4.2. An S-derivation D is linear at some $x_0 \in M$ iff there is a local basis $\{E'_m\}$ in which the components of D along every vector field vanish at x_0.

Proof. Let $\{x^i\}$ be local coordinates in a neighbourhood of x_0 and D be linear at x_0 , i.e. $W_X(x_0) = \Gamma_k X^k(x_0)$ for some Γ_k. We search for $\{E'_m = A^i_m \partial/\partial x^i\}$ in which $W'_X(x_0) = 0$. Due to (2) this is equivalent to $\Gamma_k A(x_0) + \partial A/\partial x^k|_{x_0} = 0$. Choosing $A(y)$ as in lemma 4.1 we find $A(x_0) = B$, $\partial A/\partial x^k|_{x_0} = -\Gamma_k$B. Hence $\Gamma_k A(x_0) + \partial A/\partial x^k|_{x_0} \equiv 0$ for all A defined above, i.e. the set of vector fields $\{E'_m = A^i_m \partial/\partial x^i\}$ with $[A^i_m] = A$ have the needed property. By lemma 4.1 there exists a neighbourhood U of x_0 such that $\{E'_m|_y\}$ is a basis at every $y \in U$. Hence $\{E'_m\}$ form a field of normal bases in U.

Conversely, let $W'_X(x_0) = 0$ in some $\{E'_m\}$ and every X. At x_0 from (2) we get $W_X(x_0) = -X(A)|_{x_0} A^{-1}(x_0) = \Gamma_k X^k(x_0)$ for $\Gamma_k = -E_k(A)|_{x_0} A^{-1}(x_0)$. □

From equation (2) we see that the normal frames at a given point are obtained from one another by linear transformations whose matrices are such that the action of the basic vectors on them vanish at the given point.

It follows from the definition of the torsion (see section 2) that if for an S-derivation there is a local holonomic normal frame at x_0, then its torsion is zero at x_0. Conversely, if the torsion vanishes at x_0 and normal frames exist, then all of them are holonomic at x_0.

Due to (3) the linear connections are derivations which are linear at every point at which they are defined. Hence, by proposition 4.2, for any linear connection at any point there are normal frames in which $\Gamma^i_{jk} X^k = 0$ for every vector field X, so in a normal frame $\Gamma^i_{jk} = 0$. Thus, the components of a linear connection in a normal frame at a given point vanish at that point. For symmetric linear connections this is a known result [3, 4, 6], but for non-symmetric ones this is a new one established in 1992 in [9] and reestablished independently in 1995 in [12].

5. Validity of the equivalence principle

The above results are physically important in connection with the equivalence principle. According to it (see [1, 2, 13] and references therein), at least at a point, the laws of special and general relativity coincide in a suitably chosen frame.

Let us consider a gravitational theory in which locally the gravitational field strength is identified with the local components of some (S-)derivation. The equivalence principle for such a theory, when applied on some set $U \subset M$, demands the field strength to be (locally) transformable to zero on U. Mathematically this means the (local) existence of a field of basis (or bases) on U in which the components of the mentioned (S-)derivation vanish on U. As this work deals with the cases when U is a single point or a neighbourhood of a point, the following can be concluded:

(i) All gravitational theories based on spacetimes endowed with a linear connection (e.g., the general relativity [1] and the U_4 theory [2]) are compatible with the equivalence principle at any fixed spacetime point. So, at any point there exist (local) inertial frames, which are holonomic iff the connection is torsion free (as is, e.g., the case of general relativity [1]).

(ii) Any gravitational theory based on spacetime endowed with a linear connection is compatible with the equivalence principle in a neighbourhood iff the connection is flat in it. In particular, for flat linear connections for every point there exist neighbourhoods in which there exist (local) inertial frames (bases) that are holonomic iff the connection is torsion free.

(iii) The equivalence principle is important when one tries to formulate gravitational theories on the base of some (class of) derivations. Generally, this principle will select the theories based on linear connections (cf [13]).

(iv) In the above cases the minimal-coupling principle [1, 2], mathematically realizing the equivalence principle in any gravitational theory, looks alike in those gravitational theories. For instance, if there is also a metric, it can be carried out as outlined in [2]. A physical law obtained by means of the minimal coupling principle in the considered cases identically satisfies the equivalence principle as a consequence of the underlying mathematics of the corresponding gravitational theories.

These conclusions, when applied to the case of general relativity, are in full agreement with those of [10], where it is argued that the equivalence principle is a theorem in general relativity. But our results are far more general, in particular, they are valid for any gravitational theory based on linear connections with or without torsion.

6. Conclusion

The linear connections, as we have seen, are remarkable among all derivations with their property that in a number of sufficiently general cases considered here they are the only derivations admitting special bases in which their components vanish.

This formalism also seems applicable to fields different from the gravitational one, viz. at least to those described by linear connections. This suggests the idea of extending the validity of the equivalence principle outside the gravitational interaction (cf [14]).

Elsewhere this formalism will be generalized along paths or on more general submanifolds of spacetime.

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